ISRAEL JOURNAL OF MATHEMATICS, Vol. 60, No. 2, 1987

ADMISSIBILITY SPECTRA THROUGH ω_1

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ABSTRACT
Jensen showed that any countable sequence A of A-admissibles is the initial part of the admissibility spectrum of a real. We consider ω_1-long sequences, to be realized by $B \subsetneq \omega_1$. The problem is similar to finding a club subset of a stationary set. We investigate when such a B can be forced and when one is already in V.

With the realization by Platek [P] that for any real R, the least non-R-recursive ordinal (ω_1^R) is admissible, logicians began asking a variety of questions about admissibility spectra. Sacks [Sa] showed the converse: any countable admissible is realized as the first admissible relative to some real. Different proofs were later found by H. Friedman [B] and Steel [St]; Sacks's has the advantage of producing a real minimal in the hyperdegrees among all such solutions (where a hyperdegree is an equivalence class of $\equiv_h$, and $A \leqq_h B$ if $A \in L(\omega_1^B, B)$). Jensen [J] showed how to realize a countable sequence A of A-admissibles by a real. In [L] we fuse these methods to realize countable spectra with minimality at many points along the way.

Going beyond the countable, S. Friedman [F2,3] figured out when α is the first admissible $\geqq |\alpha|$ relative to $R \subset |\alpha|$. He also showed how to realize simple spectra cofinal in the ordinals by a real, using Jensen coding [F1]. By Levy absoluteness, all the problems in realizing a sequence by a real are already contained in the case of a sequence through ω_1. Affecting all the ordinals by a real seems to call for Jensen coding. Therefore the more tractable problems about the uncountable allow for uncountable solutions.

The purpose of this paper is to investigate realizing $A \subseteq \omega_1$ by $B \subseteq \omega_1$ (without collapsing ω_1, to avoid trivialities). This problem is exceedingly similar to finding a club C through A, and even yields some new information

Received August 5, 1985 and in revised form May 16, 1986

about this procedure. To force such a B or C, A must be stationary, our further hypotheses deriving solely from considerations of admissibility. The two kinds of forcing to get C, which serve different purposes, generate two for B for different purposes. In [BHK], the countability of the conditions means that the forcing partial order is ω_1-distributive, so no new bounded sets are added. The finite conditions of [AS] imply that cardinals are preserved. In this paper, if the ground model already contains a club subset of A, countable conditions seem to be the necessary tool for its building such a B. If it has no club, or if the construction fails, then finite conditions seem required to force a club tame enough for admissibles.

Notation and background

An ordinal α is *admissible* if $L_\alpha \models \mathrm{KP}$ (= ZF-Power-Replacement $+\Delta_0$ Bounding $+\Delta_0$ Comprehension $+$ Foundation for definable classes). α is $[A_1, \ldots, A_n]$-admissible ($A_i \subseteq \mathrm{ORD}$) if

$$\langle L_\alpha[A_1, \ldots, A_n], \in, A_1, \ldots, A_n \rangle \models \mathrm{KP}.$$

(For more details on admissibility, see [B].) $[A_1, \ldots, A_n]$-Adm (or, if $n = 1$, A-Adm), the admissibility spectrum of $[A_1, \ldots, A_n]$, is $\{\alpha \mid \alpha \text{ is } [A_1, \ldots, A_n]\text{-admissible}\}$. For a potential spectrum A, B realizes A to α ($\alpha = \sup A$ if not otherwise specified) if B-Adm $\cap\, \alpha = A \cap \alpha$. The proof of Jensen's theorem alluded to earlier realizes a sequence $A \subseteq A$-Adm by forcing with a definable partial order, so the construction needs only that the ground model (in effect $\sup A$) is countable:

$$L_\lambda[A] \models \text{“}A \subseteq A\text{-Adm} \wedge \sup A \text{ is countable} \rightarrow \exists R \subseteq \omega\ R \text{ realizes } A\text{”}.$$

In Theorem 3 we use a combinatorial equivalent (under ZF) of ω_1-preservation:

$$\varnothing \Vdash_{\mathscr{P}} \text{“}\omega_1^M = \omega_1^{M[G]}\text{” iff}$$

$$\forall p \in \mathscr{P}\ \forall \langle D_i, f_i \mid i \in \omega \rangle D_i \text{ a set of mutually incompatible sentences in the forcing language } \mathscr{L}(\mathscr{G}) \text{ and}$$

$$f_i : D_i \xrightarrow{1\text{-}1} \omega_1\ \exists q \leqq p\ \forall i\ |\{\varphi \in D_i \mid q \nVdash \neg\varphi\}| \leqq \omega_0.$$

To see one direction, let $\langle D_i, f_i \mid i \in \omega \rangle$ be such a sequence. In $M[G]$, let

$$f(i)=\begin{cases} f_i(d_i) & \text{if } d_i \text{ is the unique } d\in D_i M[G]\models d,\\ 0 & \text{if } \forall d\in D_i M[G]\not\models d.\end{cases}$$

These are the only cases, since D_i is an anti-chain. f is bounded by hypothesis. So the q which force such a bound are dense, and can do so only by eliminating all but countably many possibilities. Conversely, if $p\models$ "$f:\omega\to\omega_1^M$", let $D_i=\{$"$f(i)=\alpha$" $\mid \alpha<\omega_1^M\}$. Any $q\leqq p$ which bounds the D_i also bounds rng f in ω_1.

Theorem 1. *$(V=L)$ Suppose $A\subset\omega_1$, $\forall\alpha\in A$ α is A-admissible. Then A is stationary iff there is a partial order $\mathscr{P}$ such that*

$$\Vdash_{\mathscr{P}} \exists B \text{ such that } A=\{\alpha<\omega_1 \mid \alpha \text{ is } B\text{-admissible}\} \text{ and } \omega_1=\omega_1^L.$$

Proof. $\Leftarrow$ Let B be as above in some generic extension. Let $C=\{\alpha \mid L_\alpha[B]\prec L_{\omega_1}[B]\}$. $C\subseteq B$-Adm $=A$, so A contains a club. If $L\models A$ is not stationary, let $\dot{C}\in L$ witness that. Then $C\cap\dot{C}=\varnothing$, which is a contradiction.

$\Rightarrow$ This partial order will consist of proper initial segments of such a B. A condition will be a set of ordinals correct through its sup which doesn't harm anything beyond its sup. Let

$$\mathscr{P}=\{p\subseteq\omega_1 \mid p \text{ is countable}$$
$$A\cap \sup p+1 = p\text{-Adm}\cap\sup p+1$$
$$\text{if } \sup p<\alpha<\omega_1^L,\ \alpha\notin p\text{-Adm, then } \alpha\notin A\}.$$

"$\in\mathscr{P}$" is $\Delta_1(L_{\omega_1}[A])$, since the last clause is equivalent to "if $\sup p<\alpha< L-\mathrm{rk}(p)\ldots$". $q\leqq p$ iff q end-extends p; i.e., $q-p\subseteq$ ORD $-\sup p$.

If G is $\mathscr{P}$-generic, let $B=\bigcup G$.

(1) B is unbounded in ω_1: If $p\in\mathscr{P}$ and $\alpha<\omega_1$, let $\beta>\alpha$, $\sup p$, rk p be locally countable and inadmissible. Let $R\subseteq\omega$ realize $A\restriction\beta+1$ à la Jensen, $R\in L_{\beta+10}[A]$ generic over $L_\beta[A]$. Let $q=p^\wedge R$.

(2) Since B is cofinal in ω_1, by the restrictions on the conditions themselves, A realizes B.

(3) $\varnothing\Vdash\omega_1=\omega_1^L$.

If suffices to show that $\mathscr{P}$ is countably distributive. Let $\langle D_i \mid i<\omega\rangle$ be a sequence of dense subsets of $\mathscr{P}$. $\mathscr{P}\in L_{\omega_2}$, so $\langle D_i \mid i<\omega\rangle\in L_{\omega_2}$. Let $\mathscr{P}$, $\langle D_i\rangle$, $A\in H_0\prec H_1\prec\cdots\prec H_\alpha\prec\cdots\prec L_{\omega_2}$ be the canonical increasing sequence of countable elementary substructures of L_{ω_2}:

$$H_0 = \text{Skolem Hull}(\mathcal{P}, \langle D_i \rangle, A),$$

$$H_{\alpha+1} = \text{S.H.}(H_\alpha \cup \{\omega_1 \cap H_\alpha\}),$$

$$H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha.$$

(The Skolem functions choose the L-least witness: $h_\varphi(v) =$ least w if any such that $L_{\omega_2} \models \varphi(w, v)$.) Let $\pi_\alpha : H_\alpha \overset{\approx}{\to} L_{\sigma(\alpha)}$. Let $\tau(\alpha) = (\omega_1)^{L_{\sigma(\alpha)}}$ and $\gamma(\alpha) =$ least δ such that $L_\delta \models$ "$\tau(\alpha)$ is countable". Note that $\omega_1 \cap H_\alpha = \tau(\alpha) < \sigma(\alpha) < \gamma(\alpha) < \tau(\alpha+1)$. $\tau(\lambda) = \sup_{\alpha<\lambda} \tau(\alpha)$, so $\{\tau(\alpha) \mid \alpha < \omega_1\}$ is a club.

As A is stationary, let α be such that $\tau(\alpha) \in A$. Let p_α be the L-least $\pi_\alpha(\mathcal{P})$-generic over $L_{\gamma(\alpha)}[A]$. p_α hits each $\pi_\alpha(D_i)$. But if $p \in \mathcal{P} \cap H_\alpha$, $\pi(p) = p$, so p_α hits each D_i. It remains only to check that p_α is a condition.

If $\tau(\alpha) < \beta \leqq \gamma(\alpha)$, then p_α is set-generic over $L_\beta[A]$, so it will preserve all such admissibles. If $\beta > \gamma(\alpha)$ then $p_\alpha \in L_\beta[A]$, again preserving admissibility. p_α's spectrum is correct to $\tau(\alpha)$, by the definition of $\mathcal{P}$. $\tau(\alpha) \in A$ by choice of α, so we must show that $L_{\sigma(\alpha)} \models$ "$\pi_\alpha(\mathcal{F})$ preserves the admissibility of $\tau(\alpha)$". Equivalently, we must show that (in V) $\mathcal{P}$ preserves the admissibility of ω_1.

Suppose $p_0 \Vdash \forall x \in \omega\ \exists y\, \varphi(x, y)$ ($\varphi\Delta_0$). Let $\langle p_{n+1}, y_{n+1}, z_{n+1} \rangle$ be the $L[A]$-least set such that

$$z_{n+1} \text{ witnesses that } p_{n+1} \in \mathcal{P},$$

$$p_{n+1} \leqq p_n,$$

$$\sup p_{n+1} > \operatorname{rk} y_{n+1}, \operatorname{rk} p_n, \operatorname{rk} \varphi,$$

$$L_{\operatorname{rk} p_{n+1}}[p_{n+1}] \models \varphi(n, y_{n+1}).$$

Let $p_\omega = \bigcup p_n$, $\alpha_\omega = \sup p_\omega$. p_ω's spectrum is correct to α_ω. This construction is $\Sigma_1(L_{\alpha_\omega}[A])$, so α_ω is A-inadmissible and p_ω is correct at α_ω. $p_\omega \in L_{\alpha_\omega+1}[A]$ so it affects nothing beyond α_ω. Therefore $p_\omega \in \mathcal{P}$, $p_\omega \leqq p$, and clearly

$$p_\omega \Vdash \forall x \in \omega\ \exists y \in L_{\alpha_\omega}[G]\, \varphi(x, y). \qquad \square$$

THEOREM 2. $(V = L)\ \forall A \subseteq A\text{-Adm} \cap \omega_1$

A contains a club iff $\exists B \subseteq \omega_1, \quad A = B\text{-Adm} \cap \omega_1$.

PROOF. $\Leftarrow$ $\{\alpha \mid L_\alpha[B] \prec L_{\omega_1}[B]\} \subseteq B\text{-Adm} = A$ is a club.

$\Rightarrow$ Let $\{H_\alpha \mid \alpha < \omega_1\}$ be the canonical increasing sequence of countable elementary substructures of L_{ω_2} such that $A \in H_0$, with notation π_α,

$\tau(\alpha)$, $\gamma(\alpha)$ as in the previous theorem. As before, $\omega_1 \cap H_\alpha = \tau(\alpha) < \sigma(\alpha) < \gamma(\alpha) < \tau(\alpha+1)$, and $\tau(\lambda) = \lim_{\alpha<\lambda} \tau(\alpha)$.

Since $L_{\omega_2} \models$ "A contains a club", $H_\alpha \models$ same. Let $\bar{A} \in H_0$ be a club subset of A. $\forall \alpha$ $\pi_\alpha(\bar{A}) = \bar{A} \cap \tau(\alpha)$ and $\tau(\alpha) \in \lim \bar{A}$, so $\tau(\alpha) \in \bar{A}$. Finally, $\forall \alpha$ $\langle \sigma(\beta) \mid \beta \leqq \alpha \rangle \in L_{\sigma(\alpha)+\omega}$, as follows. Since $H_\alpha \prec L_{\omega_2}$, the definition of the sequence of hulls, evaluated in H_α (with parameter A), produces $\langle H_\beta \mid \beta \leqq \alpha \rangle$. So, evaluation in $L_{\sigma(\alpha)}$ produces $\langle \pi''_\alpha H_\beta \mid \beta \leqq \alpha \rangle$. Since transitive collapses are unique, collapsing the $\pi''_\alpha H_\beta$'s results in the $L_{\sigma(\beta)}$'s. Taking hulls and collapsing are definable operations, so the result is in $L_{\sigma(\alpha)+\omega}$.

Let $\mathscr{P}$ be as in the previous theorem. $\mathscr{P}_\alpha =_{\text{def}} \mathscr{P}^{H_\alpha} = \mathscr{P} \cap L_{\sigma(\alpha)}$ because conditions p are countable sequences of countable ordinals, so $\pi_\alpha(p) = p$. Build B inductively:

Stage 0: Let p_0 be the L-least generic for $\mathscr{P}_0$ over $L_{\gamma(0)}$.

Stage $\alpha+1$: Let $p_{\alpha+1}$ be the L-least generic for $\mathscr{P}_{\alpha+1}$ over $L_{\gamma(\alpha+1)}$ through p_α.

Stage λ: Let $p_\lambda = \bigcup_{\alpha<\lambda} p_\alpha$.

Let $B = p_{\omega_1}$.

It suffices to show inductively that $p_\alpha \in \mathscr{P}_{\alpha+1}$. First let α be 0 or a successor. p_α is correct through its supremum $\tau(\alpha)$, as $\tau(\alpha) \in A$ and $\mathscr{P}$ preserves admissibility. For $\tau(\alpha) < \beta \leqq \gamma(\alpha)$, p_α is set generic over $L_\beta[A \cap \tau(\alpha)]$, so if β is p_α-inadmissible then β is A-inadmissible. $p_\alpha \in L_{\gamma(\alpha)+1}$ so it does not affect admissibility beyond $\gamma(\alpha)$. Finally, $\gamma(\alpha)+1 < \tau(\alpha+1)$ so $p_\alpha \in H_{\alpha+1}$.

For λ a limit, inductively p_λ is correct to $\tau(\lambda) = \sup_{\alpha<\lambda} \sigma(\alpha)$. We show that p_λ is $\mathscr{P}_\lambda$-generic over $L_{\sigma(\lambda)}$. Suppose $D \in H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ is dense in $\mathscr{P}_\lambda$. $D \in H_\alpha$ for some α, and $L_{\sigma(\alpha)} \models$ "$\pi_\alpha(D)$ is dense in $\mathscr{P}_\alpha$". $p_{\alpha+1} \cap \pi_\alpha(D) \neq \varnothing$, and $\pi_\alpha(D) = D \cap L_{\sigma(\alpha)}$ since $\pi_\alpha \upharpoonright \mathscr{P} = \mathrm{Id}$. $\pi_\alpha(D) \subseteq \pi_\lambda(D)$, so $p_\lambda \cap \pi_\lambda(D) \neq \varnothing$, and p_λ is generic over $L_{\sigma(\lambda)}$. As above, $\tau(\lambda) \in A$ is p_λ-admissible since $\mathscr{P}$ preserves admissibility; if $\tau(\lambda) < \beta \leqq \sigma(\lambda)$ then p_λ is set-generic over $L_\beta[A]$ and preserves A-admissibles. Finally, since $\langle \sigma(\beta) \mid \beta \leqq \lambda \rangle, A \cap \tau(\lambda) \in L_{\sigma(\lambda)+\omega}$, p_λ is definable shortly beyond $\sigma(\lambda)$, and so does not affect admissibility. □

The previous proofs used $V = L$ implicitly, in that the needed parameters were hidden. They hinge on $\mathrm{Col} \subseteq \omega_1$ which collapses each $\alpha < \omega_1$ to be countable, and a club $\bar{A} \subseteq A$. Col and $\bar{A}$ must not destroy the admissibility of $\alpha \in A$-Adm, since the approximations to B are defined using them and must preserve members of A. We indicate that some restriction on the ground model is necessary by giving an example of $A \subseteq A$-Adm containing a club but not being realized by any $B \subseteq \omega_1$. Then we force to realize any stationary A, but using conditions quite different from the earlier ones.

EXAMPLE. $A \subseteq \omega_1^V$ contains a club, but there is no $B \subseteq \omega_1^V$ such that $A = B\text{-Adm} \cap \omega_1^V$.

Let $A = \text{Adm} \cap (\omega_1^L, \omega_2^L)$. V will be $L[G]$, where G will be a (generic) minimal collapse of ω_1^L (see [N]). A condition will be an ω_1-splitting tree of finite conditions for the collapse of ω_1. That is, let

$$\begin{aligned}\mathcal{P} = \{p \mid\ & \text{dom } p = {}^{<\omega}\omega_1 \\ & \text{rng } p \subseteq \text{Levy partial order to collapse } \omega_1 \\ & \text{dom}(p(\sigma^\wedge\alpha)) = \text{dom}(p(\sigma^\wedge\beta)) \\ & \sigma \supseteq \tau \Rightarrow p(\sigma) \supseteq p(\tau) \\ & \sigma \perp \tau \Rightarrow p(\sigma) \perp p(\tau)\}.\end{aligned}$$

Let G be $\mathcal{P}$-generic. G is constructibly equivalent to $\bigcap_{p\in G}[p]$ = the unique path through all $p \in G$, which is an unbounded function $f: \omega \to \omega_1^L$. Also, all other cardinals are preserved.

G is a minimal collapse in that if $H \in L[G]$ and $L[H] \models$ "ω_1^L is countable", then $G \in L[H]$. To see this, let h be a term for a collapse of ω_1^L in $L[G]$. Let $p \in \mathcal{P}$. We will describe a fusion sequence from p, $p = p_0 \geqq p_1 \geqq \cdots$, such that $p_\omega \Vdash$ "$G \in L[H]$".

Given p_n, let $\sigma \in \text{dom } p_n$ have length n. We will define $n_\alpha \in \omega$ and $h(n_\alpha) \in \omega_1$ such that $\alpha > \beta \Rightarrow h(n_\alpha) > h(n_\beta)$, inductively on α.

Let $(q)_\tau$ be such that $(q)_\tau(\sigma) = q(\tau^\wedge\sigma)$. Extend $(p_n)_{\sigma^\wedge\alpha}$ to $\overline{p_\alpha}$ forcing a value for $h(n_\alpha)$ (for some $n_\alpha \in \omega$) greater than each $h(n_\beta)$, $\beta < \alpha$. Since each $n_\alpha \in \omega$, there must be an $m_0 \in \omega$ such that for ω_1-many α, $m_0 = n_\alpha$. Similarly, we have $m_1 \in \omega$ such that for ω_1-many such α's, dom $\overline{p_\alpha}(0) = m_1$. Let $(p_{n+1})_{\sigma^\wedge\gamma} = \overline{p_\alpha}$, where α is the γth such ordinal (i.e., $n_\alpha = m_0$ and dom $\overline{p_\alpha}(0) = m_1$). Let p_ω be the fusion of the p_n's : $p_\omega(\sigma) = p_{|\sigma|}(\sigma)$.

$p_\omega \Vdash$ "$G \in L[H]$", because at a split in p_ω each extension corresponds to different facts about h. Therefore h can tell which path the actual generic G went through.

In $V[G]$, if $A = B\text{-Adm} \cap \omega_1$, then $L[B] \models$ "ω_1^L is countable". By the minimality of the collapse, there is an α, $\omega_1^L < \alpha < \omega_2^L$, such that $G \in L_\alpha[B]$. Once we show that Adm/G-Adm is unbounded in ω_2^L we will have reached a contradiction, by the definitions of A and B.

Adm/G-Adm is unbounded in ω_2^L by density considerations. Let $p \in \mathcal{P}$, $\alpha < \omega_2^L$. Let $\beta > \alpha$, rk p be admissible. In L_β, there is an isomorphism f between p and the full tree Id : ${}^{<\omega}\omega_1^L \to {}^{<\omega}\omega_1^L$, $f(p(\sigma)) = \sigma$. Let $X \subseteq \omega_1^L$ code a well-ordering of type β. X can be coded into Id as follows. Let $g : {}^{\omega}\omega_1^L \leftrightarrow \omega_1^L$ be a bijection $\Delta_1(L_{\omega_1^L})$, $\hat{X} = \{g(X \restriction \alpha) \mid \alpha < \omega_1^L\}$. Thin Id to q so that rng$(q) = {}^{<\omega}\hat{X}$.

Pull back q to p', a thinning of p, via f^{-1}. If G is $\mathscr{P}$-generic and $p' \in G$, then X is $\Delta_1(L_{\omega_1^L}[p, G])$, so $X \in L_\beta[G]$ and $\beta \notin G$-Adm.

THEOREM 3. (ZF) *Suppose $A \subseteq A$-Adm $\cap\, \omega_1$. A is stationary iff $\exists\, \mathscr{P}$ s.t.*

$$\varnothing \Vdash_{\mathscr{P}} \text{“}\exists B\, A = B\text{-Adm} \cap \omega_1 \wedge \omega_1^{V[G]} = \omega_1^{V}\text{”}.$$

PROOF. $\Leftarrow$ As in Theorem 1.

$\Rightarrow$ This partial order would be the forcing from [AS] to produce a club C using finite conditions (essentially properties 1–3 below), were it not for the additional consideration of admissibility. Even though we must end with a club subset of A, it cannot preserve the admissibility of every point of A. (Consider its ωth member.) Also, the construction will not destroy the admissibility of $\lambda \in \lim C \cap [C, A]$-Adm, even though in general $\lambda \notin A$. So $\lambda \in \lim C$ will be required to be in A when and only when $\lambda \in C$-Adm. This is the intent of 4; 5 provides enough room to expand dom(p) while preserving 4. Furthermore, our context of admissibility theory necessitates a proof of admissibility preservation, which includes techniques unnecessary in [AS].

We begin by preparing the ground model, by forcing an A-admissibility preserving collapse of each $\alpha < \omega_1$. Let $\mathscr{P}_\alpha = \{p \mid \text{dom } p \subseteq \text{Adm} \cap \alpha \text{ is finite}$ and $p(\beta)$ is a condition in the Levy collapse of β to $\omega\}$. If $\beta < \alpha$ then (the Boolean completion of) $\mathscr{P}_\beta$ is a complete subalgebra of (the completion of) $\mathscr{P}_\alpha$. This implies that $\mathscr{P}_\alpha$ preserves relativized admissibles: if $X \in V$, $\alpha \in X$-Adm, and G is $\mathscr{P}_\alpha$ generic over $L_\alpha[X]$, then $\alpha \in [G, X]$-Adm. (For a detailed proof, see e.g. [J].) Also $\mathscr{P}_{\omega_1}$ satisfies the c.c.c.: if D is a maximal anti-chain, let $D \in H \prec H(\omega_2)$, H countable; $\pi(D) = D \cap \mathscr{P}_\alpha$, where $\alpha = \omega_1 \cap H$ and π is the transitive collapse of H; $\pi(D)$ remains a maximal anti-chain in each $\mathscr{P}_\beta$, $\beta > \alpha$, so $\pi(D) = D$. In particular, $\mathscr{P}_{\omega_1}$ is proper, so A remains stationary in a generic extension. Let G be $\mathscr{P}_{\omega_1}$-generic. $\forall \alpha < \omega_1\ L_{\alpha+10}[G] \vDash \alpha$ is countable.

Let $\mathscr{P}$ be the set of finite functions $p : \omega_1 \to \omega_1$ satisfying the following:

(1) $p(\alpha) \geqq \alpha$.

(2) If $\alpha_0 < \alpha_1$, $\alpha_i \in \text{dom } p$, then $\alpha_1 - \alpha_0 \leqq p(\alpha_1) - p(\alpha_0)$ (where $\alpha - \beta = \gamma$ iff $\beta + \gamma = \alpha$).

(3) Let fs(α) (the final segment of α) be the least γ such that γ is the order type of a final segment of α. Note that fs($\alpha + 1$) = 0.
fs($p(\alpha)$) $\geqq$ fs(α).

(4) Let $\lambda \in \text{sc}$ (λ is sufficiently closed) iff $\lambda \in A$ and $L_{\hat\lambda}[A] \vDash \lambda$ is a regular cardinal. ($\hat\lambda$ is the least p.r. closed ordinal $> \lambda$.)
If $p(\alpha) = \lambda \in A$-Adm, then α is a limit iff $\lambda \in \text{sc}$, and in this case $\alpha = \lambda$.

(5) If $\beta < \alpha \in \operatorname{dom} p$, $\beta \in \text{Adm/sc}$, then $\beta + \alpha \leqq p(\alpha)$.

$q \leqq p$ iff $q \supseteq p$.

Let C be $\mathcal{P}$-generic.

LEMMA. (i) $\operatorname{dom} C = \omega_1^L$.

(ii) $\operatorname{rng} C$ *is closed*.

(iii) $\omega_1^{V[G,C]} = \omega_1^V$.

(iv) $\alpha \in A\text{-Adm} \Rightarrow \alpha \in [C, G, A]\text{-Adm}$.

Given this lemma, the proof follows easily. Work in $L_{\omega_1}[C, G, A]$. Let $\langle g_\alpha \mid \alpha < \omega_1 \rangle = \lim C$. To build B, at stage $\alpha + 1$ choose the $L[C, G, A]$-least real which corrects the spectrum in the interval $(g_\alpha, g_{\alpha+1}]$, and code it in $(g_\alpha, g_\alpha + \omega)$. (At stage 0, correct $(0, g_0]$ and code it into ω.) At limit stages take unions. If $g_\beta < \alpha \in A$-Adm then $B \restriction g_\beta \in L_\alpha[C, G, A]$. (We use here that G collapses ordinals fast, so the correcting reals show up soon.) So if $\gamma \in (g_\alpha, g_{\alpha+1}]$ then whether $\gamma \in B$-Adm is determined by $B \cap (g_\alpha, g_\alpha + \omega)$. Hence B's spectrum is correct on all intervals $(g_\alpha, g_{\alpha+\omega})$. At λ a limit, suppose $g_\lambda \in A$-Adm. By (4), $g_\lambda \in A$. By (iv), $g_\lambda \in [C, G, A]$-Adm. $B \restriction g_\lambda$ is $\Delta_1(L_{g_\lambda}[C, G, A])$, so $g_\lambda \in B$-Adm.

PROOF OF LEMMA. We omit the ordinal arithmetic involved in verifying properties (1)–(5) when it is routine.

(i) $\operatorname{dom} C = \omega_1^L$.

Case I. $\alpha > \operatorname{dom} p$

Let $p' = p \cup \{\langle \alpha, \operatorname{rng} p + \alpha \cdot 2 \rangle\}$.

Case II. $\beta_0 < \alpha < \beta_1$, $\alpha \leqq p(\beta_0)$

Let $p'(\alpha) = p(\beta_0) + (\alpha - \beta_0)$.

Property (5): If $\beta < \beta_0$, $\beta + \alpha = \beta + \beta_0 + (\alpha - \beta_0) \leqq p(\beta_0) + (\alpha - \beta_0) = p(\alpha)$.

If $\beta = \beta_0$, $\beta_0 + \alpha = \beta_0 + \beta_0 + (\alpha - \beta_0) \leqq$ (by 3) $p(\beta_0) + (\alpha - \beta_0) = p(\alpha)$.

If $\beta_0 < \beta < \alpha$, $\beta - \alpha \leqq$ (by Case II) $p(\beta_0) + \alpha = p(\alpha)$ (as $\alpha - \beta_0 = \alpha$).

Case III. $\beta_0 < \alpha < \beta_1$, $p(\beta_0) < \alpha$

Let $\bar{\beta} = \sup\{\beta \leqq \alpha \mid \beta \in \text{Adm}\}$.

subcase A: If $\bar{\beta} \notin \text{Adm}$, let $p'(\alpha) = p(\beta_0) + (\alpha - \beta_0)$.

To verify (5) note that if $\beta < \alpha$ is admissible, then $\beta < \bar{\beta}$, so $\beta + \alpha = \alpha$.

subcase B: If $\bar{\beta} \in \text{Adm}$, let $p'(\alpha) = \max\{\bar{\beta} + \alpha,\ p(\beta_0) + (\alpha + \beta_0)\}$.

To verify (2), we must check that $\beta_1 - \alpha \leqq p(\beta_1) - (\bar{\beta} + \alpha)$. By (5) applied to p, $\bar{\beta} + \beta_1 \leqq p(\beta_1)$.

(ii) rng C is closed.

If $\gamma < p(\lambda) \in \lim$, we want λ', γ' such that $\gamma < \gamma' < p(\lambda)$ and $p \cup \{\langle \lambda', \gamma' \rangle\} \leqq p$.

An ordinal λ' is sufficiently large for the domain if $\lambda' > \operatorname{dom} p \restriction \lambda$ and o.t. $[\lambda', \lambda) = \operatorname{fs}(\lambda)$. γ' is sufficiently large for the range if $\gamma' > \operatorname{rng} p \restriction \lambda$ and o.t. $[\gamma', p(\lambda)) = \operatorname{fs}(p(\lambda))$.

Let λ' be any sufficiently large successor ordinal. Let γ' be a sufficiently large successor ordinal, which is also large enough to satisfy (1), (2), and (5). Then $p \cup \{\langle \lambda', \gamma' \rangle\} \leqq p$.

(iii) ω_1 is preserved as a cardinal.

Use the combinatorial version of ω_1-preservation from the introduction. Let $\langle D'_i, f'_i \mid i < \omega \rangle$ be a sequence of anti-chains and injections to ω_1, and $p \in \mathscr{P}$. If D'_i is not maximal, replace it with $D'_i \cup$ "$\forall \varphi \in D'_i \neg \varphi$". Replace each $\varphi \in D'_i$ by a maximal anti-chain D_φ forcing φ; let $D_i = \bigcup\{D_\varphi \mid \varphi \in D'_i\}$. Since $|\mathscr{P}| = \omega_1$, the f'_i can be converted uniformly to $f_i : D_i \xrightarrow{1\text{-}1} \omega_1$. Bounding the D_i will also bound the D'_i.

Let p, $\langle D_i \rangle$, $A \in H_0 \prec H_1 \prec \cdots \prec H_\alpha \prec \cdots \prec H(\omega_2)$ (= hereditarily ω_1-sized sets) be an elementary chain of countable models of length ω_1. Since A is stationary, there is an α such that ORD $H_\alpha \cap \omega_1 \in A$. Let

$$\pi : H_\alpha \leftrightarrow X, \quad q = p \cup \{\langle \omega_1^X, \omega_1^X \rangle\} \leqq p.$$

For all i, $\pi(D_i)$ is countable. If $d \in D_i$ is compatible with q, so is $d \restriction \omega_1^X$. $X \models$ "$d \restriction \omega_1^X$ is compatible with some $d_i \in \pi(D_i)$", since D_i is a maximal anti-chain. $\pi^{-1}(d_i) = d_i$, so d_i is compatible with d. Since D_i is an anti-chain, $d_i = d$, so $d \in D_i^X$.

(iv) $\mathscr{P}$ preserves all A-admissibles.

If $\lambda \in A$-Adm and $\lambda \neq p(\alpha)$ for all limits α, then $C \restriction \lambda$ is set-generic over $L_\lambda[G, A]$, hence preserves admissibility. (Recall that A-Adm = $[G, A]$-Adm.)

If $\lambda \in A$-Adm and $\lambda = p(\alpha)$, $\alpha \in \lim$, then $\alpha = \lambda$ and λ is sufficiently closed. Also, $C \restriction \lambda$ is $\mathscr{P}^{L_\lambda[G,A]}$-generic over $L_\lambda[G, A]$. Suppose $p \Vdash$"$L_\lambda[C, G, A] \models \forall x \, \exists y \, \varphi(x, y)$", rk p, rk $\varphi < \beta_0 < \lambda$. We will extend p to force "$L_\lambda[C, G, A] \models \forall x^{\beta_\omega} \, \exists y^{\beta_\omega} \varphi(x, y)$", for some β_ω such that $\beta_0 < \beta_\omega < \lambda$.

Let $\langle q, y \rangle_{\bar{x}, \bar{p}}$ be the $L[G, A]$-least set such that $q \leqq \bar{p}$, rk $q >$ rk y, $q \Vdash \varphi(\bar{x}, y)$. Let β_{n+1} be the least admissible $\geqq \sup\{\operatorname{rk}\langle q, y \rangle_{\bar{x}, \bar{p}} \mid \bar{x} \in \mathscr{T}_{\beta_n}, \bar{p} \in \mathscr{P}_{\beta_n}, \bar{p} \leqq p\}$. Let $\beta_\omega = \lim \beta_n$, $\bar{p} = p \cup \{\langle \beta_\omega, \beta_\omega \rangle\}$.

We need only show (a) $\bar{p} \in \mathscr{P} \cap L_\lambda$ and (b) $\bar{p} \Vdash$"$L_\lambda[C, G, A] \models \forall x^{\beta_\omega} \exists y^{\beta_\omega} \varphi(x, y)$". The most important point is that this definition can be equally well evaluated in $L_\lambda[GA]$, or even $L_{\beta_\omega}[G, A]$. Proving such a fact needs that $\Vdash$ reflects: roughly, $p \Vdash \psi$ iff $p \upharpoonright \mathrm{rk}\, \psi \Vdash \psi$. This is the point of the next few lemmas, due essentially to Steel [St].

Let $\langle \gamma_\nu \mid \nu < \omega_1 \rangle$ enumerate the countable p.r. closed ordinals.

DEFINITION. $q_0 \sim_{\gamma_\nu} q_1$ if $q_i(\alpha) < \gamma_\nu \Rightarrow q_i(\alpha) = q_{1-i}(\alpha)$, and if α is the least ordinal $< \gamma_\nu$ such that $q_i(\alpha) \geqq \gamma_\nu$, then $\alpha \in \mathrm{dom}\, q_{1-i}$.

EXTENSION LEMMA. *If* $q_0 \sim_{\gamma_\nu} q_1$, $\nu' < \nu$, $r_0 \leqq q_0$, *then* $\exists r_1 \leqq q_1$, $r_0 \sim_{\gamma_{\nu'}} r_1$.

PROOF. If $\mathrm{rng}\,(r_0/q_0) \upharpoonright \gamma_{\nu'} \subseteq \gamma_{\nu'}$, let $r_1 = q_1 \cup r_0 \upharpoonright \gamma_{\nu'}$. Otherwise, let α be the least ordinal $< \gamma_{\nu'}$ such that $(r_0/q_0)(\alpha) \geqq \gamma_{\nu'}$. Let $r_1 = q_1 \cup r_0 \upharpoonright \alpha \cup \{\langle \alpha, \min(r_0(\alpha), \gamma_\nu + \alpha) \rangle\}$.

RETAGGING LEMMA. *If* $\mathrm{rk}\, \varphi < \nu$, $q_0 \sim_{\gamma_\nu} q_1$, *then* $q_0 \Vdash \varphi$ *iff* $q_1 \Vdash \varphi$.

PROOF. This is a straightforward induction, using the extension lemma for the negation case.

FORCING LEMMA. *If* $\tau_\nu = \nu$, *then* $\Vdash \upharpoonright L_\nu[G, A] \times L_\nu[G, A]$ *is* $\Delta_1(L_\nu[G, A])$.

PROOF. The very definition of $\Vdash$ restricted is a straightforward $\Delta_1(L_\nu[G, A])$ induction, except for the negation case. Let p, φ be such that $\mathrm{rk}\, p$, $\mathrm{rk}\, \varphi < \nu$. Let $\nu' = \max(\mathrm{rk}\, p, \mathrm{rk}\, \varphi) + 1$. If $p \Vdash \neg\varphi$, then $\forall r \leqq p\ r \nVdash \varphi$, and in particular $\forall r \leqq p$ such that $r \in L_{\gamma_{(\nu'+2)}}[G, A]\ r \nVdash \varphi$. Otherwise, $\exists r_0 \leqq p\ r_0 \Vdash \varphi$. Let $r_1 \leqq p$ be as given in the proof of the extension lemma, for $p = q_0 = q_1$ and $p \sim_{\gamma_{(\nu'+1)}} p$. $r_1 \Vdash \varphi$ by the retagging lemma, and $r_1 \in L_{\gamma_{(\nu'+2)}}[G, A]$. So we can eliminate the unbounded quantifier in "$p \Vdash \neg\varphi$" by using as the definition "$\forall r \in L_{\gamma_{(\nu'+2)}}[G, A]\ r \leqq p \rightarrow r \nVdash \varphi$".

(a) Properties (1) and (3) are clear. β_ω is a limit of admissibles, so (5) and (2) are clear. To show (4), it suffices to show that β_ω is A-inadmissible. $\gamma_{\beta_\omega} = \beta_\omega$, so the forcing relation restricted to $L_{\beta_\omega}[G, A]$ is Δ_1. Therefore, the definition of β_ω is $\Delta_1(L_{\beta_\omega}[G, A])$, so β_ω is A-inadmissible.

$\beta_\omega < \lambda$ because λ is s.c.

(b) We need to show that $\forall \bar{q} \leqq \bar{p}\ \forall \bar{x} \in \mathscr{T}_{\beta_\omega}\ \exists \bar{r} \leqq \bar{q}\ \exists \bar{y} \in \mathscr{T}_{\beta_\omega}, \bar{r} \Vdash \varphi(\bar{x}, \bar{y})$.

Let $q = \bar{q} \upharpoonright \beta_\omega$, $q \in \mathscr{P}_{\beta_n}$, $\bar{x} \in \mathscr{T}_{\beta_n}$ some n. Therefore $\exists r \leqq q$, $r \in \mathscr{P}_{\beta_{n+1}}$, $\bar{y} \in \mathscr{T}_{\beta_{n+1}}$ such that $r \Vdash \varphi(\bar{x}, \bar{y})$. Let $\bar{r} = r \cup \bar{q}$. $\bar{r} \leqq r$, so $\bar{r} \Vdash \varphi(x, y)$, and $\bar{r} \leqq \bar{q}$. □

QUESTION. It seems that building a B when possible requires countable conditions, while in general forcing such a B requires finite conditions. Is there some way to make this precise and to prove it?

Acknowledgments

The contents of this paper formed part of the author's doctoral thesis. He would like to thank Professor Sy Friedman for his capable advising, and the NSF and MIT for their financial support. He also acknowledges the referee, whose correction of an earlier draft led to the reorganization of the paper.

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